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1987 J. Phys. A: Math. Gen. 20 L67

(http://iopscience.iop.org/0305-4470/20/2/005)

J. Phys. A: Math. Gen. **20** (1987) L67–L70. Printed in the UK

LETTER TO THE EDITOR

Large-N spontaneous magnetisation in zero dimensions

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Received 27 October 1986

Abstract. A two-parameter model of Hermitian matrices in zero-dimensional space is solved in the large-N limit. A very interesting phase diagram and a spontaneous magnetisation are exhibited.

Field theoretical models with matrix valued field variables display interesting and unusual properties in the limit of infinite order of the matrices. Indeed it was recently shown [1, 2] that such models exhibit a third-order phase transition, for sufficiently negative values of the squared mass, even in zero dimensions of spacetime.

In this letter we study the large-N limit of the matrix model with quartic interaction and a linear coupling to a constant external field. This limit, though in a zero-dimensional model, is a sort of thermodynamic limit for the volume of the internal symmetry group and leads to effects similar to those of the usual thermodynamic limit for systems in two or three space dimensions. It is then likely that these investigations will provide insights into many statistical mechanics models and into the Goldstone and Goldstone–Higgs mechanisms in quantum field theory.

We consider the partition function

$$Z(m^2, g, h) = \lim_{N\to\infty} \int d^{N^2}M \exp\{-[\mathrm{Tr}(\tfrac{1}{2}m^2M^2 + (g/N)M^4 - h\sqrt{N}M)]\} \times\left(\int d^{N^2}M \exp[-(\tfrac{1}{2}|m^2|\mathrm{Tr}\, M^2)]\right)^{-1} \tag{1}$$

where M is a Hermitian $N\times N$ matrix, g is a positive number and m^2 and h are real numbers. Since g may be chosen equal to one, without loss of generality, the model actually depends on the two dimensionless variables $\alpha \equiv \frac{1}{2}hg^{-1/4}$, $\beta \equiv \frac{1}{2}m^2g^{-1/2}$. In the large-$N$ limit, the spectral function $u(\lambda)$, that describes the eigenvalue density, solves the saddle point equation

$$⨍_L d\mu \frac{u(\mu)}{\lambda-\mu} = \tfrac{1}{2}m^2\lambda + 2g\lambda^3 - \tfrac{1}{2}h \tag{2}$$

with $\lambda \in L$, the unknown support of $u(\lambda)$. The Green functions $G_p(m^2, g, h) \equiv \langle \mathrm{Tr}(M^p)\rangle$ are given by the moments

$$G_p(m^2, g, h) = \int_L d\lambda\, \lambda^p u(\lambda). \tag{3}$$

With the single-segment ansatz $L_1 = [A, D]$, (2) is easily inverted by standard methods [3]. We find

$$u(\lambda) = \frac{g}{\pi}[(D-\lambda)(\lambda-A)]^{1/2}[2\lambda^2 + (A+D)\lambda + \tfrac{1}{2}(A+D)^2 + \tfrac{1}{4}(D-A)^2 + m^2/2g]. \tag{4}$$

0305-4470/87/020067+04$02.50

The extrema A and D are determined by the two coupled equations for $\sigma \equiv A+D$ and $\delta \equiv D-A$, which follow by requiring that the Green function generator

$$F(\lambda) \equiv \int_{L_1} \mathrm{d}\mu \frac{u(\mu)}{\lambda-\mu} \qquad \lambda \notin L_1 \tag{5}$$

behaves as $1/\lambda$ for large values of $|\lambda|$. They are

$$2g\sigma^3 + 3g\sigma\delta^2 + 2m^2\sigma - 4h = 0 \tag{6}$$

$$3g\delta^4 + 12g\delta^2\sigma^2 + 4m^2\delta^2 - 64 = 0. \tag{7}$$

The density $u(\lambda)$ is non-negative definite provided one of the following conditions is satisfied:

$$C_1(\sigma, \delta) \equiv 3\sigma^2 + 2\delta^2 + 4m^2/g \geqslant 0 \qquad \text{if } A \leqslant -\tfrac{1}{4}\sigma \leqslant D \tag{8}$$

$$C_2(\sigma, \delta) \equiv 6\sigma^2 + 3\delta^2 + 6\sigma\delta + 4\beta \geqslant 0 \qquad \text{if } -\tfrac{1}{4}\sigma \geqslant D \tag{9}$$

$$C_3(\sigma, \delta) \equiv 6\sigma^2 + 3\delta^2 - 6\sigma\delta + 4\beta \geqslant 0 \qquad \text{if } -\tfrac{1}{4}\sigma \leqslant A. \tag{10}$$

The condition $C_1(\sigma, \delta) = 0$ defines a curve in the (α, β) plane given by the equation

$$2(g\beta^2 - 20)^{3/2} = 4\beta(11\beta^2 - 60) - 135\alpha^2 \tag{11}$$

whereas the curve corresponding to $C_2(\sigma, \delta) = 0$ is conveniently expressed in the parametric form

$$\beta = -2(t^2 - 2t + 2)\left(\frac{3}{t^3(4-t)}\right)^{1/2} \tag{12a}$$

$$\alpha = -2(3t - 2)\left(\frac{4}{3t^3(4-t)}\right)^{3/4} \tag{12b}$$

with $0 < t \leqslant \frac{3}{2}$. The third curve is obtained from the second with the exchange $\alpha \to -\alpha$ (figure 1).

Equations (4), (6) and (7) define a unique solution, hereafter called the weak coupling solution, in the region of the (α, β) plane outside the shaded area and above

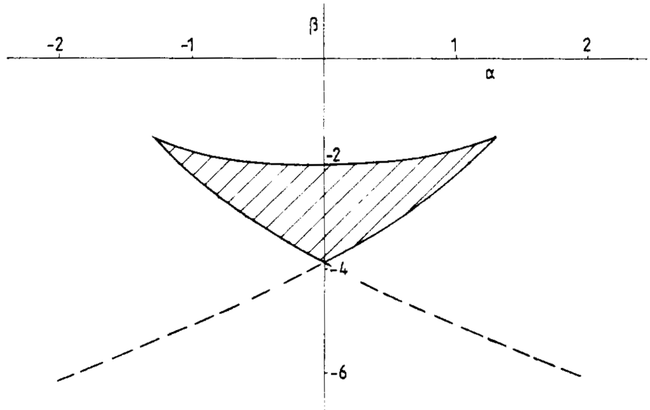

Figure 1. The phase diagram of the model in terms of the two dimensionless parameters. The full curve represents the border between the weak and the strong coupling solutions.

the broken curves. In the shaded area there are no positive solutions for such equations, while in the region below the broken curves there are two positive solutions. One is the analytic continuation of the unique solution from the region above and it leads to a first-order phase transition along the $\alpha = 0$ line, if $\beta < -\sqrt{15}$. The other solution, for $\alpha > 0$, may be regarded as the analytic continuation of the weak coupling solution with $\alpha < 0$ across the $\alpha = 0$ line and it represents an unstable physical solution.

Then for (α, β) in the shaded area, we consider a two-segment ansatz $L_2 = [A, B] \cup [C, D]$, with $A \leq B \leq C \leq D$. This choice leads to the evaluation of several definite elliptic integrals. Nonetheless, simplifications occur in all the relevant quantities which may be written in terms of elementary functions. We find

$$u(\lambda) = (g/\pi)\phi(\lambda)(2\lambda + A + B + C + D) \tag{13}$$

where $\phi(\lambda) = \pm[(D-\lambda)(\lambda - C)(\lambda - B)(\lambda - A)]^{1/2}$ and the upper (lower) sign holds for $\lambda \in [C, D]$ $(\lambda \in [A, B])$. The asymptotic condition on $F(\lambda)$ now implies the three equations:

$$AB + AC + AD + BC + BD + CD = \tfrac{3}{4}S^2 + m^2/2g \tag{14}$$

$$ABC + ABD + ACD + BCD = \tfrac{1}{2}S^3 + (m^2/2g)S + h/2g \tag{15}$$

$$ABCD = \tfrac{5}{16}S^4 + \tfrac{3}{8}(m^2/g)S^2 + (h/2g)S + m^4/16g^2 - 1/g \tag{16}$$

where $S = A + B + C + D$. The requirement $u(\lambda) \geq 0$, $\lambda \in L_2$, implies $B \leq -\frac{1}{2}S \leq C$. However we verify that the saddle point equation (2), with the density $u(\lambda)$ given in (13)-(16), is still satisfied at $\lambda = -\frac{1}{2}S$. Then $-\frac{1}{2}S$ is B or C, and therefore $\phi(-\frac{1}{2}S) = 0$, i.e.

$$15S^4 + 12(m^2/g)S^2 + 12(h/g)S + (m^4/g^2 - 16/g) = 0. \tag{17}$$

Equation (17), essential for the determination of the solution, makes the Green function generator stationary with respect to variations of the baricentre $\frac{1}{4}S$ of the support L_2.

The set of equations (13)-(17) allows two solutions in the shaded area (and none outside it): the physical one, hereafter called the strong coupling solution, is smoothly connected to the weak coupling solution across the boundary, thus resulting in a high-order phase transition on the boundary and a first-order phase transition along the segment $\alpha = 0$, $-\sqrt{15} < \beta < -2$. The second solution, for positive α, is the analytic continuation of the strong coupling solution for negative α across the $\alpha = 0$ line and it represents an unstable solution.

It is now possible to evaluate the magnetisation $M = G_1(g, m^2, h)$

$$M = \int_{L_1} d\lambda\, \lambda u(\lambda) = g\frac{\sigma\delta^2}{64}(6\sigma^2 + 3\delta^2 + 2m^2/g) \tag{18}$$

for the weak coupling solution and

$$M = \int_{L_2} d\lambda\, \lambda u(\lambda) = -\tfrac{3}{16}gS^5 - \tfrac{1}{4}m^2S^3 - \tfrac{3}{8}hS^2 + (1 - m^4/16g)S - m^2h/8g \tag{19}$$

for the strong coupling solution. In the limit of vanishing external field, a non-trivial magnetisation appears for $\beta < -2$. The behaviour of M is described in figure 2.

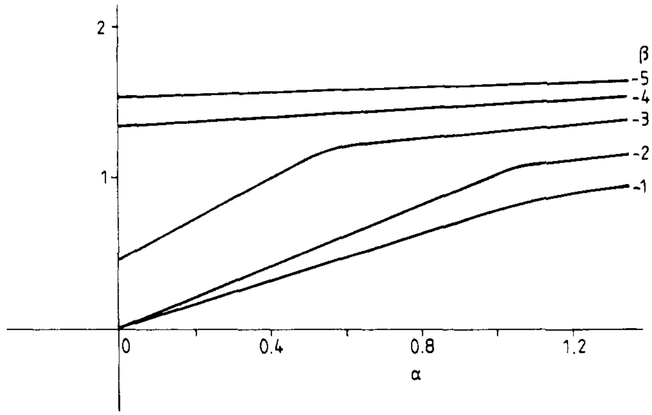

Figure 2. Magnetisation curves plotted against the external field, for various negative values of the dimensionless parameter β. Spontaneous magnetisation develops for $\beta < -2$.

A related model, with complex matrices and a different coupling to the external fields, was recently studied [2] by solving the recursive Schwinger-Dyson equations in the large-N limit. A transition curve in the parameter space was found, where the free energy has a third-order phase transition, quite analogous to our equation (11). However, the method did not lead to the evaluation of the density $u(\lambda)$. The boundary between the two phases found in [2] corresponds to the square of our equation (11), after taking into account the different coupling to the external field. It seems that the further conditions, analogous to (9) and (10), were missed. Our finding that the weak coupling solution holds up to infinite coupling casts some doubt on the conclusions of that letter.

We thank F Riva and P Butera for useful discussions.

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